

Stable Ranks, K -Groups and Witt Groups of some Banach and C^* -Algebras *

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Abstract : We show that certain dense and spectral invariant subalgebras of a C^* -algebra have the same bilateral Bass stable rank. This is a partial answer for (a version of) an open problem raised by R.G. Swan. Then, for certain Banach algebras, we indicate when the homotopy groups $\pi_i(GL_n(A))$ stabilize for large n . This is an improvement of a result due to G. Corach and A. Larotonda. Using some results due to M. Karoubi, we show the isomorphism of the Witt group of a symmetric Banach algebra with the K_0 -group of its enveloping C^* -algebra. The question if this is true for all involutive Banach algebras was raised by A. Connes.

Résumé : On démontre que certaines sous-algèbres denses et pleines d'une C^* -algèbre ont le même rang stable (de Bass) bilatère. Ceci est une réponse partielle à un problème ouvert de R.G. Swan. Pour certaines algèbres de Banach A , les groupes d'homotopie $\pi_i(GL_n(A))$ sont stables pour n assez grand. Ceci

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est une amélioration d'un résultat de G. Corach et A. Laro-
tonda. En utilisant des résultats dus à M. Karoubi, on démontre
l'isomorphisme du groupe de Witt d'une algèbre de Banach symétrique
avec le groupe K_0 de son C^* -algèbre enveloppante. La question
de savoir si cet isomorphisme a lieu pour toutes les algèbres de
Banach involutives à été soulevée par A. Connes.

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1 Introduction

There are several notions of stable ranks for topological algebras. We will discuss here some of them for some Banach and C^* -algebras and their incidence in the study of certain groups related to K -Theory. Their definitions and main properties are surveyed in the next section.

The problems we are dealing with are

- Swan's problem for subalgebras of C^* -algebras and for the bilateral Bass stable rank ;
- stabilization of the homotopy groups of the general linear group of a Banach algebra ;
- the isomorphism between the Witt group and the K_0 -group of the enveloping C^* -algebra of a given symmetric Banach algebra.

We present now the motivation for these problems and the results we will prove.

1.1 Swan's problem for the bilateral Bass stable rank

The density theorem in K -Theory implies that a Banach dense and spectral invariant subalgebra A of a Banach algebra B has the same K -theory as B , that is the inclusion morphism $j : A \rightarrow B$ induces isomorphisms $j_* : K_i(A) \rightarrow K_i(B)$, $i = 0, 1$. Recall that A is called *spectral invariant* in B if $a \in A$ with a invertible in B imply that a is invertible in A . Spectral invariant Banach subalgebras are closed under the holomorphic functional calculus, that is, for $a \in A$ and g holomorphic in a neighborhood of the spectrum of a in B , the element $g(a)$ of B lies in A .

It will be recalled in the next section that stable ranks stabilize the K -groups. The question whether dense spectral invariant subalgebras have the same stable rank arises. It was Richard G. Swan [Sw, p.206] who raised this question for the Bass stable rank and for the projective stable rank. In this note we consider Swan's problem for a variant of the Bass stable rank, called bilateral Bass stable rank [Ba].

Swan's problem for several stable ranks was considered in [Ba]. A consequence of the main result there gives a positive answer of Swan's problem for the Bass stable rank in the case when B is a C^* -algebra and A is a dense and spectral invariant $*$ -subalgebra, which is a Fréchet Q -algebra (cf. [Ba] for definitions) in its own topology and closed under C^∞ -functional calculus of selfadjoint elements.

We give in this note a sufficient condition for $*$ -subalgebras of C^* -algebras to have the same bilateral Bass stable rank. In particular, we prove that dense spectral invariant (D_p) and $*$ -subalgebras of C^* -algebras have the same bilateral Bass stable rank. The notion of (D_p) -subalgebras of Banach algebras was recently considered by Kissin and Shulman [KiSh2].

1.2 Computing homotopy groups

Stability theorems for the K -groups $K_0(A)$ and $K_1(A)$ in terms of the Bass stable rank are stated in the next section. Moreover, if the Bass stable rank of A is finite, the homotopy groups $\pi_i(GL_n(A))$ stabilize for large n . Here $GL_n(A)$ is the group of invertible elements of $M_n(A)$, the set of all $n \times n$ matrices with entries in A .

To be more specific, it was proved by G. Corach and A.R. Larotonda [CoLa1] that the map

$$\pi_i(GL_{n-1}(A)) \rightarrow \pi_i(GL_n(A))$$

between the homotopy groups is surjective for $n \geq Bsr(A) + i + 1$ and injective for $n \geq Bsr(A) + i + 2$. For other results of this type we refer to [Ri2], [Th], [Sc1], [Sc2], [Zh1], [Zh2].

We prove here a stability result in terms of the connected stable rank and the general stable rank which is a slight improvement of the result of Corach and Larotonda.

The following consequence for commutative Banach algebra is obtained. Suppose that A is unital and commutative. Then the map

$$\pi_i(GL_{n-1}(A)) \rightarrow \pi_i(GL_n(A))$$

is surjective for $n \geq Bsr(A) + [i/2] + 2$ and injective for $n \geq Bsr(A) + [(i + 1)/2] + 2$.

1.3 Computing the Witt groups

We will consider in this section symmetric Banach algebras, commutative or not. Recall that an involutive Banach algebra is said to be *symmetric* if every element of the form x^*x has spectrum included in positive real closed half-line. The symmetry is equivalent in the commutative case to $m(x^*) = \overline{m(x)}$ for all characters m .

The following problem has been raised by A. Connes in his book [Co].

It is known (see [Co]) that for C^* -algebras A one has an isomorphism between the Witt group $W_0(A)$ and the K_0 -group $K_0(A)$ of K -theory. We refer to the next section for definitions. In [Co] it is asked if, for an involutive Banach algebra A , the Witt group $W_0(A)$ is isomorphic to $K_0(C^*(A))$, the K_0 -group of the enveloping C^* -algebra of A . J.-B. Bost (cf. [Co]) proved this for *commutative* involutive A . We show in the present note that an affirmative answer (even for higher Witt groups) follows for symmetric Banach involutive algebras from the work of M. Karoubi [Ka2].

1.4 Acknowledgments

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2 Background

A Banach algebra will mean a complex Banach algebra. The unity is always denoted by e .

2.1 Stable ranks

For a Banach algebra A we denote by $Lg_n(A)$ the set of all n -tuples $(a_1, \dots, a_n) \in A^n$ with the property $Aa_1 + \dots + Aa_n = A$, i.e. generating A as a left ideal. By definition, the (left) *Bass stable rank* $Bsr(A)$ of A is the smallest positive integer n such that the following condition

$$(Bsr)_n \quad \text{for every } (a_1, \dots, a_{n+1}) \in Lg_{n+1}(A), \text{ there exists } (c_1, \dots, c_n) \in A^n \text{ such that } (a_1 + c_1a_{n+1}, \dots, a_n + c_na_{n+1}) \in Lg_n(A),$$

holds, or infinity if no such number n exists. Note that in some of the earlier papers the notion of the Bass stable rank was defined using different indexing conventions. The condition $(Bsr)_n$ was devised by H. Bass in order to determine values of n for which every matrix in $GL_n(A)$ can be row reduced by addition operations with coefficients from A to a matrix with the same last row and column as the identity matrix and to obtain stability results in K -theory.

It can be proved that $(Bsr)_n$ implies $(Bsr)_{n+1}$ ([Va],[Kr]) and that the Bass stable rank of A equals the stable rank of the opposed algebra A° ([Va],[Wa]). That is, the left Bass stable rank defined above is equal to the right Bass stable rank which can be defined in a similar way using the set $Rg_n(A)$ of all n -tuples (a_1, \dots, a_n) generating A as a right ideal. If A is non-unital, we define the Bass stable rank of A as the Bass stable rank of the algebra A_+ obtained by adjoining a unit element to A .

A variant of the Bass stable rank was introduced in [Ba]. We call *bilateral Bass stable rank* $bBsr(A)$ of the unital Banach algebra A the smallest positive integer n such that the following condition holds for $k \geq n$

$$(bBsr)_k \quad \text{for every } (a_1, \dots, a_{k+1}) \in Lg_{k+1}(A), \text{ there exist } (c_1, \dots, c_k) \in A^k, (d_1, \dots, d_k) \in A^k \text{ such that } (a_1 + c_1a_{k+1}d_1, \dots, a_k + c_ka_{k+1}d_k) \in Lg_k(A),$$

or infinity if no such number exists. We have $bBsr(A) \leq Bsr(A)$, with equality for commutative A . There are [Ba] C^* -algebras A with $bBsr(A) \neq Bsr(A)$.

M.A. Rieffel [Ri1] introduced the notion of topological stable rank as follows : the (left) *topological stable rank* $tsr(A)$ of A is the smallest positive integer n such that $Lg_n(A)$ is dense in A^n , or infinity if no such number exists. If $Lg_n(A)$ is dense in A^n , then $Lg_m(A)$ is dense in A^m for every $m \geq n$. A symmetric notion, the right topological stable rank $rtsr$ can be defined by considering the set $Rg_n(A)$ instead of $Lg_n(A)$. The left and the right topological stable ranks coincide for Banach algebras with a continuous involution. It is an open question [Ri1, Question 1.5] if $tsr(A)$ equals $rtsr(A)$ for all Banach algebras A . If A is non-unital, we define the topological stable rank of A as the topological stable rank of A_+ . We refer to [Ri1] for several properties of tsr .

2.2 Other stable ranks

We mention briefly other notions of stable ranks. The *connected stable rank* $csr(A)$ of the Banach algebra A is [Ri1] the least integer n such that $GL_k(A)_0$ acts transitively (by left multiplication) on $Lg_k(A)$ for every $k \geq n$, or, equivalently [Ri1], the least integer n such that $Lg_k(A)$ is connected for every $k \geq n$. We put $csr(A) = \infty$ if no such n exists. This notion is left-right symmetric [CoLa2]. Recall that $GL_k(A)_0$ is the connected component of $GL_k(A)$ containing the identity.

The *left (right) general stable rank* of A is defined [Ri1] as the smallest integer n such that $GL_k(A)$ acts on the left (right) transitively on $Lg_k(A)$ for all $k \geq n$. If no such integer exists we set $gsr(A) = \infty$. For C^* -algebras it is related to the cancellation property for finitely generated projective A -modules. For instance, the right general stable rank is the smallest positive integer n such that $W \oplus A \cong A^k$ for some $k \geq n$ implies $W \cong A^{k-1}$, whenever W is a finitely generated projective left A -module. This notion is left-right symmetric [CoLa2] and we will denote by $gsr(A)$ the common value.

2.3 Properties of stable ranks

It was proved in [Ba] that condition $(bBsr)_n$ holds for A if and only if every onto unital algebra morphism $f : A \rightarrow B$, B a Banach algebra, induces an onto mapping $f_n : Lg_n(A) \rightarrow Lg_n(B)$. A similar characterization holds for the (left) Bass stable rank by replacing onto algebra morphisms from A with onto module morphisms of left A -modules from ${}_A A$, which is A viewed as a left A -module. Also [Ba], $tsr(A) \leq n$ if and only if for every $\varepsilon > 0$ and

every $(a_1, \dots, a_{n+1}) \in Lg_{n+1}(A)$, there exists $(c_1, \dots, c_n) \in A^n$ such that $(a_1 + c_1 a_{n+1}, \dots, a_n + c_n a_{n+1}) \in Lg_n(A)$ and $\|c_i a_{n+1}\| \leq \varepsilon$ for $i = 1, \dots, n$. This yields $bBsr(A) \leq Bsr(A) \leq tsr(A)$ for all Banach algebras A . For C^* -algebras we have $Bsr(A) = tsr(A)$ as was shown by Herman and Vaserstein [HeVa]. For the unital commutative C^* -algebra $C(X)$ we have

$$bBsr(C(X)) = Bsr(C(X)) = tsr(C(X)) = [(\dim X)/2] + 1,$$

where $\dim X$ is the Čech-Lebesgue covering dimension of X [Pe].

We have

$$gsr(A) \leq csr(A) \leq 1 + Bsr(A) \leq 1 + tsr(A) .$$

2.4 Stable ranks and K -theory

The Bass and topological stable ranks are useful for stability results in the K -theory of topological algebras. We state here some of them in terms of Bsr ; since $Bsr(A) \leq tsr(A)$, we can replace in these statements the Bass stable rank with the topological one. It was proved by G. Corach and A.R. Larotonda [CoLa1] that, for a Banach algebra A with unit e , the map

$$GL_n(A)/GL_n(A)_0 \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix} \in GL_{n+1}(A)/GL_{n+1}(A)_0$$

is bijective for $n \geq 1 + Bsr(A)$. Thus the topological K_1 -group $K_1(A)$ of A , defined as the direct limit of $GL_n(A)/GL_n(A)_0$ under these inclusions, stabilizes if $Bsr(A)$ is finite. A similar result can be stated for the (topological) K_0 group of a Banach algebra. Indeed, $K_0(A)$ can be written [Ka1] as the direct limit of $\bar{P}_{2n}(A)$ under the inclusions

$$p \mapsto \begin{pmatrix} p & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

where $\bar{P}_{2n}(A)$ is the quotient of $P_{2n}(A)$, the subset of idempotents of $M_{2n}(A)$, given by the equivalence relation

$$p_1 \approx p_2 \Leftrightarrow \exists \alpha \in GL_{2n}(A), \quad \alpha p_1 \alpha^{-1} = p_2 .$$

Then the map $\bar{P}_{2n}(A) \rightarrow \bar{P}_{2n+2}(A)$ is bijective for $n \geq Bsr(A)$ [Cor].

2.5 Witt groups

Let A be an involutive unital Banach algebra. In Hermitian K -theory we denote by $L_0(A)$ the Grothendieck group of finitely generated, projective A -modules endowed with non-degenerate quadratic forms. In order to define higher L -groups $L_n(A)$, we consider the orthogonal group $O_{k,k}(A)$ of isometries of A^{2k} endowed with the standard hyperbolic form. It can be described as the group of $2k \times 2k$ matrices X of the form

$$X = \begin{pmatrix} M & N \\ P & Q \end{pmatrix},$$

where $M, N, P, Q \in M_n(A)$ and $X'X = XX' = I_{2n}$. Here X' is the following matrix

$$\begin{pmatrix} {}^tQ^* & {}^tN^* \\ {}^tP^* & {}^tM^* \end{pmatrix},$$

while I_{2n} is the identity matrix.

Let $O(A)$ be the inductive limit of $O_{k,k}(A)$ under the inclusions

$$a \rightarrow \begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}.$$

The topological Hermitian K -groups $L_n(A)$ are defined as homotopy groups of $O(A)$, namely

$$L_n(A) = \pi_{n-1}(O(A)).$$

We refer to [Ka2] and the references therein for further information. In particular, the hyperbolic functor induces a homomorphism $K_n(A) \rightarrow L_n(A)$.

The n -th Witt group $W_n(A)$ is defined as

$$W_n(A) = \text{Coker}(K_n(A) \rightarrow L_n(A)).$$

For $n = 0$ we obtain the classical Witt group W_0 .

3 Computing $bBsr$ for dense subalgebras of C^* -algebras

The following is the announced partial result for Swan's problem for the bilateral Bass stable rank.

Theorem 3.1. *Let $(B, \|\cdot\|_0)$ be a unital C^* -algebra and let $(A, \|\cdot\|_0)$ be a dense and spectral invariant $*$ -subalgebra containing the unity of B . Let p be a fixed positive integer. Suppose that there is a family $\{\|\cdot\|_i\}_{1 \leq i \leq p}$ of norms on A and positive constants C_i , $1 \leq i \leq p$, such that*

$$\|x^2\|_i \leq C_i \|x\|_i \|x\|_{i-1}, \quad 1 \leq i \leq p,$$

for any selfadjoint $x = x^ \in A$. Suppose also that $(A, \|\cdot\|_p)$ is a Banach algebra. Then $bBsr(A) = bBsr(B)$.*

Proof : We have [Ba, Theorem 4.7(i)] $bBsr(A) \leq bBsr(B)$. For the reverse inequality, suppose $bBsr(A) = n$ and let $\mathbf{b} = (b_1, \dots, b_{n+1}) \in Lg_{n+1}(B)$. We will prove that \mathbf{b} is *bilateral reducible* in B , that is there exist (c_1, \dots, c_n) , (d_1, \dots, d_n) in B^n such that

$$(b_1 + c_1 b_{n+1} d_1, \dots, b_n + c_n b_{n+1} d_n) \in Lg_n(B).$$

Let J be the closed two-sided ideal in B generated by b_{n+1} . It was proved in [Ba, Theorem 4.7(ii)] that \mathbf{b} is bilateral reducible in B if $J \cap A$ is dense in J . We prove now, using several techniques borrowed from [KiSh1, KiSh2], that $\overline{J \cap A} = J$ holds for all two-sided closed ideals of B .

Let $x = x^*$ be a selfadjoint element of A . Let t be a real number. Let k be the positive integer such that $2^{k-1} < |t| \leq 2^k$. Set $y = itx/2^k$. Let

$$M = \max_{1 \leq j \leq p} \{C_j, \|\exp(y)\|_j\}.$$

Then $\|\exp(y)\|_0 = 1$ and

$$\|\exp(y)\|_j = \|\exp(itx/2^k)\|_j \leq \exp(|t|\|x\|_j/2^k) \leq M$$

for $j = 1, \dots, p$. Using [KiSh2, Lemma 1] we get

$$\begin{aligned} \|\exp(itx)\|_p &= \|\exp(2^k y)\|_p = \|\exp(y)^{2^k}\|_p \\ &\leq \|\exp(y)\|_0^{2^k - S(k,p)} \prod_{j=0}^{p-1} \|\exp(y)\|_{p-j}^{a(k,j)} C_{p-j}^{a(k,j+1)} \\ &= \prod_{j=0}^{p-1} \|\exp(y)\|_{p-j}^{a(k,j)} C_{p-j}^{a(k,j+1)} \leq M^b, \end{aligned}$$

where

$$a(k, j) = \binom{k}{j} \quad \text{and} \quad b = \sum_{j=0}^{p-1} [a(k, j) + a(k, j+1)] .$$

For $k > 2p$ (and so for all t such that $|t|$ is sufficiently large) we have (cf. [KiSh2, p. 416]) :

$$\|\exp(itx)\|_p \leq M|2t|^{\psi(t)},$$

where

$$\psi(t) = \frac{(\log_2 |2t|)^{p-1} \log_2 M}{(p-1)!}.$$

This implies that

$$\int_{-\infty}^{+\infty} \frac{\log \|\exp(itx)\|_p}{1+t^2} dt < +\infty.$$

It follows [KiSh2] from Shilov's [Na, §15.6] condition of regularity that A is locally normal in B . This means [KiSh1] that there is a commutative Banach $*$ -subalgebra $B(x)$ in B such that e and x belong to $B(x)$ and such that $A(x) = A \cap B(x)$ is a dense normal subalgebra of $B(x)$. It was proved in [KiSh1, Theorem 13] that if J is a closed two-sided ideal in B , then $J \cap A$ is dense in J . This completes the proof. \diamond

The condition

$$\|x^2\|_i \leq C_i \|x\|_i \|x\|_{i-1}, \quad 1 \leq i \leq p,$$

for selfadjoint x in the above theorem is a particular case of the condition (D_p) studied in [KiSh2]. They introduced (D_p) -subalgebras A of Banach algebras $(B, \|\cdot\|_0)$ as dense subalgebras of B for which there exist norms $\{\|\cdot\|_i\}_{1 \leq i \leq p}$ and positive constants D_i , $1 \leq i \leq p$, such that $(B, \|\cdot\|_p)$ is a Banach algebra and

$$(D_p) \quad \|xy\|_i \leq D_i (\|x\|_i \|y\|_{i-1} + \|x\|_{i-1} \|y\|_i), \quad x, y \in B, \quad 1 \leq i \leq p.$$

Thus the condition in Theorem 3.1 is obtained from condition (D_p) for $y = x$ with $C_j = 2D_j$. We obtain the following consequence.

Corollary 3.2. *Dense spectral invariant (D_p) and $*$ -subalgebras of C^* -algebras have same bilateral Bass stable rank.*

We refer to [KiSh2] for several examples of D_p -subalgebras of C^* -algebras. For instance, the differential subalgebras of order p studied by Blackadar and Cuntz [BlCu] are (D_p) -subalgebras and, for $p = 1$, these classes coincide.

The above Corollary can be viewed as a noncommutative generalization of a result due to Vaserstein [Va]. He proved that the Banach algebras $C^k(X)$ of k -differentiable functions on a compact manifold X have the same Bass stable rank as $C(X)$. It is well-known that C^* -algebras are noncommutative analogies of the algebras of continuous functions. (D_p) -subalgebras of C^* -algebras can be viewed [KiSh2] as noncommutative analogies of algebras of smooth functions. Note also that for commutative Banach algebras the bilateral Bass stable rank coincide with the usual Bass stable rank.

4 Computing homotopy groups

Denote by \mathbf{S}^i the i -dimensional sphere in \mathbb{R}^{i+1} .

The proof of the following stabilization theorem is based upon [CoLa1] and [Ri2].

Theorem 4.1. *Let A be a unital Banach algebra and let $n \geq \text{csr}(A)$. The canonical morphism from $\pi_i(GL_{n-1}(A))$ into $\pi_i(GL_n(A))$ is*

- a) *surjective for $n \geq \text{csr}(C(\mathbf{S}^i, A))$*
- b) *injective for $n \geq \text{gsr}(C(\mathbf{S}^{i+1}, A))$.*

In particular (cf. [CoLa1]), the canonical morphism is

- a) *surjective for $n \geq \text{Bsr}(A) + i + 1$*
- b) *injective for $n \geq \text{Bsr}(A) + i + 2$.*

Moreover, if A is commutative, then the canonical morphism from $\pi_i(GL_{n-1}(A))$ into $\pi_i(GL_n(A))$ is

- a) *surjective for $n \geq \text{Bsr}(A) + [i/2] + 2$*
- b) *injective for $n \geq \text{Bsr}(A) + [(i+1)/2] + 2$.*

Here $[\cdot]$ is the integer part.

Proof : By [CoLa2], the map $T : GL_n(A) \rightarrow Lg_n(A)$, $T(X) = Xe_n$, ($e_n = (0, \dots, 0, 1)$) is a Serre fibration and, with the same proof, the map $T' : GL_n(A) \rightarrow Lc_n(A)$, $T'(X) = Xe_n$, is also a Serre fibration. Here $Lc_n(A)$ denotes the space of last columns of invertible $n \times n$ matrices with entries in A :

$$Lc_n(A) = \{Me_n : M \in GL_n(A)\} .$$

The stability subgroup of e_n induced by the action of $GL_n(A)$ under $Lg_n(A)$ and $Lc_n(A)$ consists of matrices whose last column is e_n , that is matrices of the form

$$\begin{pmatrix} x & 0 \\ c & e \end{pmatrix} ,$$

where $x \in GL_{n-1}(A)$ and c is an arbitrary row in A of length $n-1$. Viewing $GL_{n-1}(A)$ as a subset of $GL_n(A)$ via the embedding

$$a \rightarrow \begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix} ,$$

we obtain a deformation retract of the stability subgroup of e_n onto $GL_{n-1}(A)$ by carrying the off-diagonal entry c linearly to zero. Since T' is a Serre fibration, one has the homotopy exact sequence [Sp, Ch. 7]

$$\rightarrow \pi_{i+1}(Lc_n(A)) \rightarrow \pi_i(GL_{n-1}(A)) \rightarrow \pi_i(GL_n(A)) \rightarrow \pi_i(Lc_n(A)) \rightarrow .$$

This long exact sequence ends with [Sp]

$$\pi_0(GL_{n-1}(A)) \rightarrow \pi_0(GL_n(A)) \rightarrow \pi_0(Lc_n(A))$$

viewed as pointed sets. The base points in the groups are taken to be their identity elements, while the base points in $Lg_n(A)$ and $Lc_n(A)$ are taken to be e_n .

For $n \geq csr(A)$ one has [Ba] $n \geq gsr(A)$ and $n \geq Lccsr(A)$. Here $Lccsr(A)$, the *last columns connected stable rank* is, by definition [Ba], the least integer k such that for all $n \geq k$ the set $Lc_n(A)$ is connected. Since [Ba]

$$Lc_n(A)_0 = Lg_n(A)_0 = GL_n(A)_0 e_n ,$$

we have $Lc_n(A) = Lg_n(A)$ and $\pi_0(Lc_n(A))$ is trivial. From the proof of [CoLa1, Theorem 6.2] it follows that there is a bijection

$$\pi_i(Lg_n(A)) \rightarrow \pi_0(Lg_n(C(\mathbf{S}^i, A))) .$$

Since $n \geq csr(C(\mathbf{S}^i, A))$, we get that $\pi_i(Lc_n(A)) = \pi_i(Lg_n(A))$ is also trivial. From the long exact sequence above we obtain the surjectivity of

$$\pi_i(GL_{n-1}(A)) \rightarrow \pi_i(GL_n(A)) .$$

It follows again from the homotopy exact sequence that the map

$$\pi_i(GL_{n-1}(A)) \rightarrow \pi_i(GL_n(A))$$

is injective if and only if the map

$$\pi_{i+1}(GL_n(A)) \rightarrow \pi_{i+1}(Lc_n(A))$$

is surjective. Suppose that $n \geq gsr(C(\mathbf{S}^{i+1}, A))$ and thus the map

$$GL_n(C(\mathbf{S}^{i+1}, A)) \rightarrow Lg_n(C(\mathbf{S}^{i+1}, A))$$

is surjective. Let $f : \mathbf{S}^{i+1} \rightarrow Lg_n(A)$ be a map preserving the base point $f(1) = e_n$, and so representing an element of $\pi_{i+1}(Lg_n(A))$. Then f can be identified with an element of $Lg_n(C(\mathbf{S}^{i+1}, A))$ such that $f(1) = e_n$. By the surjectivity, there is $g \in GL_n(C(\mathbf{S}^{i+1}, A))$ such that $g(t)e_n = f(t)$, $t \in \mathbf{S}^{i+1}$. In particular, we have $g(1)e_n = e_n$ and thus $g(1)^{-1}e_n = e_n$. Then the map

$$h : \mathbf{S}^{i+1} \ni t \rightarrow h(t) = g(t)g(1)^{-1} \in GL_n(C(\mathbf{S}^{i+1}, A))$$

is such that $h(t)e_n = f(t)$ and $h(1) = e$. Since $Lg_n(A) = Lc_n(A)$, we obtain that the map

$$\pi_{i+1}(GL_n(A)) \rightarrow \pi_{i+1}(Lc_n(A))$$

is surjective and thus

$$\pi_i(GL_{n-1}(A)) \rightarrow \pi_i(GL_n(A))$$

is injective.

To obtain the first consequence (which is the result of Corach and Laro-tonda), note that $gsr(A) \leq csr(A) \leq 1 + Bsr(A)$ and [CoLa1] $Bsr(C(\mathbf{S}^i, A)) \leq i + Bsr(A)$.

For the second consequence, we use a result due to F. D. Suarez [Su] : if A is commutative, then

$$Bsr(C(\mathbf{S}^i, A)) \leq [i/2] + 1 + Bsr(A) .$$

This completes the proof. \diamond

It was conjectured that inequality $Bsr(C(\mathbf{S}^2, A)) \leq 1 + Bsr(A)$ might be true [Ri1], [Su], at least for C^* -algebras or for commutative Banach algebras. Suppose it holds for every complex Banach algebra. Then an inductive argument implies that $Bsr(C(\mathbf{S}^i, A)) \leq [(i+1)/2] + Bsr(A)$ for every i . This would lead to an improvement of Theorem 4.1.

5 Computing higher Witt groups

The aim of this section is to show how some results due to M. Karoubi [Ka2] yield an answer to the above-mentioned question of A. Connes in the case of symmetric Banach algebras. In fact, the following statement for higher Witt groups is true.

Theorem 5.1. *Let A be a symmetric Banach $*$ -algebra. For all $n \geq 0$, $L_n(A)$ is isomorphic in a natural way to $K_n(A) \oplus K_n(A)$. Therefore*

$$W_n(A) \simeq K_n(A) \simeq K_n(C^*(A)) .$$

Proof : The first statement was proved by M. Karoubi [Ka2, Theorem 2.3] for the so-called C-algebras. As was proved by H. Leptin [Le] and J. Wichmann [Wi], if A is a symmetric Banach $*$ -algebra, then the matrix algebra $M_n(A)$ is also symmetric. It follows that symmetric Banach $*$ -algebras are C-algebras. The fact that $K_n(A) \simeq K_n(C^*(A))$ follows from the density theorem in K -theory. Indeed, since A is symmetric, A is dense and spectral invariant subalgebra of $C^*(A)$. Indeed, $a \in A$ has the same spectrum in A or in $C^*(A)$. \diamond

For symmetric Banach $*$ -algebras of finite stable rank we can give a better description of the Witt groups of A in terms of $C^*(A)$. We recall that $\bar{P}_{2n}(A)$ is the quotient of $P_{2n}(A)$, the subset of idempotents of $M_{2n}(A)$, given by the equivalence relation $p_1 \approx p_2 \Leftrightarrow \exists \alpha \in GL_{2n}(A), \alpha p_1 \alpha^{-1} = p_2$.

Corollary 5.2. *Let A be a symmetric Banach $*$ -algebra. Suppose $s = tsr(A)$ is finite. Then*

$$W_n(A) \simeq \bar{P}_{2s}(C^*(A)) \text{ for } n \text{ even}$$

and

$$W_n(A) \simeq GL_s(C^*(A))/GL_s(C^*(A))_0 \text{ for } n \text{ odd.}$$

Proof : Recall A is a dense and spectral invariant subalgebra of $C^*(A)$. Then we have (cf. for instance [Ba])

$$Bsr(A) \leq Bsr(C^*(A)) = tsr(C^*(A)) \leq tsr(A) = s .$$

According to [Cor] and [Ri2], we have $K_0(C^*(A)) \simeq P_{2s}(C^*(A))$ and

$$K_1(C^*(A)) \simeq GL_s(C^*(A))/GL_s(C^*(A))_0 .$$

The result now follows from the above Theorem and Bott periodicity theorem. \diamond

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